Quantum Parafermions in the $SL(2,R)/U(1)$ WZNW Black Hole Model

C. Ford

DESY Zeuthen, Platanenallee 6, Zeuthen 15738, Germany

ford@ifh.de

Abstract

Starting with its classical parafermion algebra, we consider the quantisation of the $SL(2,R)/U(1)$ WZNW black hole model.

Key words: conformal field theory, black holes, parafermions.

1 Introduction

About ten years ago it was realised that certain $2d$ conformal field theories have a black-hole interpretation. Witten [1] pointed out that a non-nilpotently gauged $SL(2,R)$ Wess Zumino Novikov Witten (WZNW) theory describes a two dimensional Euclidean black hole. He also suggested that the theory *may be integrable* opening up the possibility of an exact quantisation of the black hole model. Gervais and Saveliev [2] went one step further. They noted that certain non-Abelian Toda theories also represent black holes. The importance of this observation is that the integrability of such (*nilpotently-gauged* WZNW) theories was not in doubt. Furthermore, even though these are $2d$ conformal field theories they can correspond to *higher* dimensional black holes. More precisely, certain nilpotently gauged B_n WZNW theories contain n-dimensional black holes. With a view to performing an exact canonical quantisation Bilal [3] considered the simplest case, $B_2 \equiv SO(5)$, in some detail. The theory can be described via the Lagrangian

$$\gamma^2\mathcal{L} = \partial_z r\partial_{\bar z} r + \tanh^2 r\partial_z t\partial_{\bar z} t + \partial_z\phi\partial_{\bar z}\phi + \cosh(2r)e^{2\phi}, \tag{1}$$

where r is a positive scalar field, t an 'angular' field in that t and $t+2\pi$ are identified and ϕ is a real scalar field. $z=\tau+\sigma$ and $\bar z=\tau-\sigma$ are light cone coordinates and γ is a coupling constant. Using a general construction of Gervais and Savaliev one can write down explicitly the general solution to the classical equations of motion. This can be recast as a canonical transformation (CT) mapping the 'physical' r, t, ϕ fields to three free fields ϕ_1, ϕ_2 , ϕ_3. The next step would be to 'promote' the CT to a quantum-mechanical operator identity exchanging physical and free fields. Such calculations are well known from Liouville theory [4, 5]. However the CT considered in [3] is *much* more complicated than its Liouville counterpart (it takes several pages just to write it down).

The Lagrangian for the $SL(2,R)/U(1)$ model is simpler

$$\gamma^2\mathcal{L} = \partial_z r \partial_{\bar{z}} r + \tanh^2 r \partial_z t \partial_{\bar{z}} t. \tag{2}$$

In fact it is just the same as the B_2 theory without the 'Liouville' ϕ field. One can associate to the Lagrangian (2) a 'target space' metric

$$ds^2 = (dr)^2 + \tanh^2 r (dt)^2. \tag{3}$$

It is this that Witten interpreted as a $2d$ Euclidean black hole. One can regard the B_2 theory as a black hole 'interacting' with Liouville 'matter'. Guided by the solution of the non-Abelian Toda theory Müller and Weigt [6] deduced the solution of the $SL(2,R)/U(1)$ model and gave a CT mapping the physical r and t fields onto two free fields ϕ_1 and ϕ_2. This work established that the theory is indeed integrable. To express the solution it is convenient to introduce the 'Kruskal' coordinates

$$u = e^{it}\sinh r, \qquad \bar{u} = e^{-it}\sinh r. \tag{4}$$

The CT is

$$u = e^{i\gamma(\phi_2+\bar{\phi}_2)}\left[e^{\gamma(\phi_1+\bar{\phi}_1)}(1+\Phi\bar{\Phi}) - \frac{1}{4}e^{-\gamma(\phi_1+\bar{\phi}_1)} + \frac{i}{2}\left(\Phi e^{\gamma(\phi_1-\bar{\phi}_1)} + \bar{\Phi}e^{-\gamma(\phi_1-\bar{\phi}_1)}\right)\right], \tag{5}$$

where the $\phi_i \equiv \phi_i(z)$ and $\bar{\phi}_i \equiv \phi_i(\bar{z})$ denote respectively the chiral and anti-chiral part of a free field. The chiral object $\Phi \equiv \Phi(z)$ is defined via the differential equation

$$\partial_z \Phi(z) = e^{-2\gamma\phi_1(z)}\partial_z\phi_2(z), \tag{6}$$

and similarly for $\bar{\Phi}(\bar{z})$.

2 Parafermion algebra

Underlying the integrability of $SL(2,R)/U(1)$ model is a set of 'parafermionic' chiral fields

$$V_\pm = \frac{1}{\gamma^2}e^{\pm i\nu}\left(\partial_z r \pm i\tanh r \partial_z t\right). \tag{7}$$

ν is defined through the differential equations

$$\partial_z\nu = (1+\tanh^2 r)\partial_z t, \qquad \partial_{\bar{z}}\nu = \cosh^{-2} r \partial_{\bar{z}} t. \tag{8}$$

The integrability condition for these equations is *one* of the equations of motion. Using the other equation of motion it follows that the $V_\pm$ are *chiral,* i.e. $\partial_{\bar{z}}V_\pm = 0$. Similarly, one can define a pair of anti-chiral parafermions. The chiral component of the energy momentum tensor has the Sugawara form

$$T(z) = \gamma^2 V_+(z)V_-(z). \tag{9}$$

It turns out that the $V_\pm(z)$ are much simpler when written in terms of the free fields

$$V_\pm(z) = \frac{1}{\gamma}\left(\partial_z\phi_1(z) \pm i\partial_z\phi_2(z)\right)\exp\left(\pm 2i\gamma\phi_2(z)\right). \tag{10}$$

The parafermions satisfy a closed Poisson bracket (PB) algebra. For concreteness let us fix the boundary conditions and the basic PB's of the free fields. We take spacetime to be $S^1 \times R$ so that we have periodicity in the spatial direction

$$u(\sigma + 2\pi, \tau) = u(\sigma, \tau). \tag{11}$$

For the free fields we have the mode expansions

$$\phi_k(z) = \frac{1}{2}q_k + \frac{z}{4\pi}p_k + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{a_n^{(k)}}{n}e^{-inz}, \quad \bar{\phi}_k(\bar{z}) = \frac{1}{2}q_k + \frac{\bar{z}}{4\pi}p_k + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{\bar{a}_n^{(k)}}{n}e^{-in\bar{z}}. \tag{12}$$

These fields are *not* periodic

$$\phi_k(z + 2\pi) = \phi_k(z) + \frac{p_k}{2}, \qquad \bar{\phi}_k(\bar{z} - 2\pi) = \bar{\phi}_k(\bar{z}) - \frac{p_k}{2}, \tag{13}$$

where the p_k are momentum zero modes. However, the *full* free fields, $\phi_i(z) + \bar{\phi}_i(\bar{z})$, are. The Fourier coefficients satisfy the PB relations

$$\{q_k, p_l\} = \delta_{kl}, \quad \{a_m^{(k)}, a_n^{(l)}\} = -im\delta_{kl}\delta_{m+n\,0}, \quad \{q_k, a_m^{(l)}\} = \{p_k, a_m^{(l)}\} = 0, \tag{14}$$

and similarly for the $\bar{a}_n^{(k)}$'s.

Note that, like the ϕ_i's, the $V_\pm$'s are not periodic. When computing PB's and their quantum counterparts we prefer to deal with periodic objects. Thus instead of the $V_\pm$'s we will consider

$$W_\pm(z) = \frac{1}{\gamma}\left(\partial_z\phi_1(z) \pm i\partial_z\phi_2(z)\right)e^{2i\gamma\varphi_2(z)}, \tag{15}$$

where φ_2 is ϕ_2 with the momentum zero mode removed and the whole zero mode of the full free field included, i.e. $\varphi_2(z) = \frac{1}{2}q_2 + \phi_2(z)|_{p_2=0}$. These periodic parafermions obey the non-linear PB algebra

$$\begin{aligned}
\{W_\pm(z), W_\pm(z')\} &= \gamma^2 W_\pm(z)W_\pm(z')h(z - z'), &(16)\\
\{W_\pm(z), W_\mp(z')\} &= -\gamma^2 W_\pm(z)W_\mp(z')h(z - z') + \frac{1}{\gamma^2}\left(\partial_z + \frac{i\gamma p_2}{2\pi}\right)\delta_{2\pi}(z - z'), &(17)\\
\{p_2, W_\pm(z')\} &= \mp 2i\gamma W_\pm(z'), &(18)
\end{aligned}$$

where

$$h(z) = \left(\epsilon_{2\pi}(z) - \frac{z}{\pi}\right), \tag{19}$$

is the periodic sawtooth function defined in terms of the (non-periodic) stairstep function $\epsilon_{2\pi}(z)$ [1]. $\delta_{2\pi}(z)$ denotes the periodic delta function. Observe that the zero mode p_2 enters into the algebra. Using (9) and the parafermion algebra it is easy to see that the parafermions have conformal weight one

$$\{T(z), W_\pm(z')\} = -\partial_{z'} W_\pm(z')\delta_{2\pi}(z-z') + W_\pm(z')\partial_z\delta_{2\pi}(z-z') \mp \frac{i\gamma p_2}{2\pi} W_\pm(z')\delta_{2\pi}(z-z'). \tag{20}$$

One can also derive the Virasoro algebra

$$\{T(z), T(z')\} = -\partial_{z'} T(z')\delta_{2\pi}(z-z') + 2T(z)\partial_z\delta_{2\pi}(z-z'). \tag{21}$$

3 Quantum parafermions

Now we turn to the quantisation of the model. One could simply go ahead and try to quantise the CT (5). In Liouville theory locality and conformal invariance (i.e. the requirement that the exponentials are primary) were sufficient to uniquely fix the form of the quantum transformation. However, we now have double the number of degrees of freedom as in the Liouville case and so locality and conformal invariance are too weak. To fix the quantum transformation we will demand that the physical fields 'close' with respect to the *parafermions* as well as the energy momentum tensor. This will be explained in more detail in the next section. It is clear that we need to develop a quantum version of the parafermion algebra given in the last section.

Our starting point for the quantisation of the parafermions will be the free field representation (15). The free fields can be quantised in the usual manner by defining commutators to be $i\hbar$ times the corresponding PB's. Let us define parafermion operators corresponding to (15) as

$$W_\pm(z) = \frac{1}{\gamma} : (\eta\partial_z\phi_1(z) \pm i\partial_z\phi_2(z))\, e^{\pm 2i\gamma\varphi_2(z)} : . \tag{22}$$

With a little bit of hindsight we have included a deformation parameter η ($\eta \to 1$ as $\hbar \to 0$). As usual the colons denote normal ordering. To effect this we decompose the chiral free fields as follows

$$\phi_i(z) = \frac{1}{2}q_i + \frac{p_i}{4\pi}z + \phi_i^+(z) + \phi_i^-(z), \tag{23}$$

where

$$\phi_i^\pm(z) = \pm\frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{a_{\pm n}^{(i)}}{n}e^{\mp inz}. \tag{24}$$

[1] $\epsilon_{2\pi}(z) = 2n+1$ for $2\pi n < z < (2n+2)\pi$ which coincides with $\mathrm{sign}(z)$ for $-2\pi < z < 2\pi$.

Normal ordering is defined by moving the ϕ^+'s to the right of the ϕ^-'s in all expressions. For the zero modes Hermitian normal ordering will be understood:

$$: e^{2q} f(p) := e^q f(p) e^q. \tag{25}$$

It now seems that all we have to do to determine the quantum parafermion algebra is to compute the commutators corresponding to the Poisson brackets (16,17,18). Unfortunately, such objects appear to be ill-defined. For example, on computing $[W_+(z), W_-(z')]$ one obtains meaningless contributions such as $f(z-z')\delta'_{2\pi}(z-z')$ where the derivative of $f(z)$ diverges at $z=0$. It is conceivable that some kind of UV renormalisation in addition to the normal ordering can 'cure' this problem. In [7] a different tack was taken. By deforming the commutator a well defined parafermion algebra was derived. The starting point was the following operator identity

$$\frac{W_+(z)W_+(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} - \frac{W_+(z')W_+(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} = \frac{i\hbar}{2\gamma^2}\left(\eta^2 - 1 + \frac{\gamma^2\hbar}{\pi}\right) : e^{2i\gamma\varphi(z)} e^{2i\gamma\varphi(z')} : \partial_z \delta_{2\pi}(z-z'), \tag{26}$$

where

$$h^\pm(z) = \frac{1}{2} h(z) \mp \log\left(4\sin^2\frac{z}{2}\right) \tag{27}$$

can be viewed as the positive and negative frequency parts of the saw-tooth function $h(z)$, respectively (see the technical remarks at the end of this section). The left hand side of (26) reduces to a commutator in the limit $\hbar \to 0$, while the right hand side is well defined. It is however not quite what we want, since the operator $: e^{2i\gamma\varphi_2(z)} e^{2i\gamma\varphi_2(z')} :$ cannot be rewritten locally in terms of the parafermions as we would expect for a closed parafermion algebra. We can remove the offending term altogether by imposing the restriction

$$\eta^2 - 1 + \frac{\gamma^2\hbar}{\pi} = 0, \tag{28}$$

so that

$$\frac{W_+(z)W_+(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} - \frac{W_+(z')W_+(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} = 0. \tag{29}$$

This is the quantum relation corresponding to the PB (16). To check this one can expand the exact formula in powers of $\hbar$

$$[W_+(z), W_+(z')] - i\hbar\gamma^2 W_+(z)W_+(z')h(z-z') + O(\hbar^2) = 0. \tag{30}$$

Here we have used $h(z) = h^+(z) + h^-(z)$ which follows immediately from (27). The quantum relations corresponding to the other brackets are

$$\frac{W_-(z)W_-(z')}{e^{i\hbar\gamma^2 h^+(z-z')}} - \frac{W_-(z')W_-(z)}{e^{-i\hbar\gamma^2 h^-(z-z')}} = 0, \tag{31}$$

$$\frac{W_+(z)W_-(z')}{e^{-i\hbar\gamma^2 h^+(z-z')}} - \frac{W_-(z')W_+(z)}{e^{i\hbar\gamma^2 h^-(z-z')}} = \frac{i\hbar}{\gamma^2}\left(\partial_z + \frac{i\gamma p_2}{2\pi}\right)\delta_{2\pi}(z-z'), \tag{32}$$

$$[p_2, W_\pm(z)] = \pm 2\hbar\gamma W_\pm(z). \tag{33}$$

As in the derivation of (29) it is necessary to impose (28) to eliminate extraneous operators.

We now turn to the energy momentum tensor. Classically this is just a simple product of the parafermions, i.e. $T(z) = \gamma^2 W_+(z) W_-(z)$. This does not make sense at the quantum level. The way out is to define $T(z)$ as a PB rather than a product. Consider

$$\begin{aligned}\{D_z W_+(z), W_-(z')\} &= \gamma^2 D_z W_+(z) W_-(z') h(z-z') - 2T(z)\delta_{2\pi}(z-z') \\ &\quad + \frac{1}{\gamma^2} D_z^2 \delta_{2\pi}(z-z'), \end{aligned} \tag{34}$$

where

$$D_z = \partial_z + \frac{i\gamma p_2}{2\pi}. \tag{35}$$

This is a perfectly good, albeit unwieldy, classical definition of $T(z)$. The point is that we know how to 'quantise' such brackets. In fact (34) is essentially the derivative of (17) whose quantum analogue is (32). A detailed calculation yields

$$\begin{aligned}\frac{D_z W_+(z) W_-(z')}{e^{-i\hbar\gamma^2 h^+(z-z')}} - \frac{W_-(z') D_z W_+(z)}{e^{i\hbar\gamma^2 h^-(z-z')}} &= \frac{i\hbar}{\gamma^2}\left(1 + \frac{\hbar\gamma^2}{2\pi}\right) D_z^2 \delta_{2\pi}(z-z') \qquad (36) \\ -2i\hbar\eta^2 \left(:(\partial_z\phi_1)^2(z'): + :(\partial_z\phi_2)^2(z'): + \frac{\hbar\gamma}{2\pi\eta}\partial_z^2\phi_1(z') \right. &+ \left. \frac{\hbar\gamma^2}{(4\pi\eta)^2}\right)\delta_{2\pi}(z-z'). \end{aligned}$$

The second entry on the right hand side corresponds to the term $-2T(z')\delta_{2\pi}(z-z')$ of the classical PB (34) suggesting the following identification

$$T(z) =: (\partial_z\phi_1)^2(z) : + : (\partial_z\phi_2)^2(z) : + \frac{\hbar\gamma}{2\pi\eta}\partial_z^2\phi_1(z) + \frac{\hbar\gamma^2}{(4\pi\eta)^2}. \tag{37}$$

This is indeed the energy-momentum tensor of a free field theory with an additional improvement term. One can check that this obeys a Virasoro algebra.

Classically the parafermions are primary fields of weight one. Quantum mechanically the commutator

$$\begin{aligned}[T(z), W_+(z')] &= i\hbar\left(1 + \frac{\hbar\gamma^2}{2\pi}\right) W_+(z')\partial_z\delta_{2\pi}(z-z') - i\hbar\partial_{z'} W_+(z')\delta_{2\pi}(z-z') \\ &\quad + \frac{\hbar\gamma}{2\pi} : p_2 W_+(z') : \delta_{2\pi}(z-z') \end{aligned} \tag{38}$$

shows that the quantum parafermions have the conformal weight $1 + \hbar\gamma^2/(2\pi)$.

We end this section with some technical details relating to the derivation of the quoted operator identities (see also the appendix of [7]). The key formulae are the commutators for the $\phi_i^\pm(z)$ entering into the decomposition (23). A straightforward calculation gives

$$[\phi_i^\pm(z), \phi_j^\pm(z')] = 0, \qquad [\phi_i^\pm(z), \phi_j^\mp(z')] = -\frac{i}{4}\hbar\delta_{ij} h^\pm(z-z'), \tag{39}$$

where

$$h^{\pm}(z) = \epsilon^{\pm}(z) - \frac{z}{2}. \tag{40}$$

Here the $\epsilon^{\pm}(z)$ denote the positive and negative frequency parts of the stairstep function, and have the Fourier representation

$$\epsilon^{+}(z) = \frac{z}{2\pi} + \frac{i}{\pi}\sum_{n>0}\frac{e^{-in(z-i\varepsilon)}}{n}, \qquad \epsilon^{-}(z) = \frac{z}{2\pi} + \frac{i}{\pi}\sum_{n<0}\frac{e^{-in(z+i\varepsilon)}}{n}. \tag{41}$$

Note that we have included a convergence factor, $\varepsilon > 0$, *which should be retained until the end of all calculations.* In the limit $\varepsilon \to 0$ (40) agrees with our earlier definition (27). We will also employ the 'split' delta functions [8] $\delta^{+}(z) = 1/2\partial_z\epsilon^{+}(z)$

$$\delta^{+}(z) = \frac{1}{4\pi} + \frac{1}{2\pi}\sum_{n>0} e^{-in(z-i\varepsilon)} = -\frac{1}{4\pi} + \frac{1}{2\pi}\frac{1}{1-e^{-i(z-i\varepsilon)}}, \tag{42}$$

and similarly $\delta^{-}(z) = 1/2\partial_z\epsilon^{-}(z)$, which have the property that $\delta^{+}(z) + \delta^{-}(z) \to \delta_{2\pi}(z)$ as $\varepsilon \to 0$.

Our parafermion operator $W_{+}(z)$ can be written

$$W_{+}(z) = e_{-}(z)\nu(z)e_{+}(z), \tag{43}$$

where

$$e_{\pm}(z) = e^{i\gamma q_2}e^{2i\gamma\phi_2^{\pm}(z)}, \quad \nu(z) = \frac{1}{\gamma}\left(\eta\partial_z\phi_1(z) + i\partial_z\phi_2(z)\right). \tag{44}$$

A simple calculation (using $e^A e^B = e^B e^A e^{[A,B]}$ for $[A,B]$ complex) gives

$$e^{-i\hbar\gamma^2 h^{+}(z-z')}W_{+}(z)W_{+}(z') = e_{-}(z)\nu(z)e_{-}(z')e_{+}(z)\nu(z')e_{+}(z'). \tag{45}$$

The right hand side is still not normal ordered; a little algebra yields

$$\begin{aligned} e^{-i\hbar\gamma^2 h^{+}(z-z')}W_{+}(z)W_{+}(z') \;=\;& :W_{+}(z)W_{+}(z'): \\ &+e_{-}(z)e_{-}(z')[\nu^{+}(z),\nu^{-}(z')]e_{+}(z)e_{+}(z') \\ &+e_{-}(z)[\nu(z),e_{-}(z')]\nu(z')e_{+}(z)e_{+}(z') \\ &+e_{-}(z)e_{-}(z')\nu(z)[e_{+}(z),\nu(z')]e_{+}(z') \\ &+e_{-}(z)[\nu(z),e_{-}(z')][e_{+}(z),\nu(z')]e_{+}(z'), \end{aligned} \tag{46}$$

where $\gamma\nu^{\pm}(z) = \eta\partial_z\phi_1^{\pm}(z) + i\partial_z\phi_2^{\pm}(z)$. Evaluating the commutators on the right hand side

$$\begin{aligned} \frac{W_{+}(z)W_{+}(z')}{e^{i\hbar\gamma^2 h^{+}(z-z')}} =&\, :W_{+}(z)W_{+}(z'): + i\hbar :\left(e^{2i\gamma\varphi_2(z)}W_{+}(z') - W_{+}(z)e^{2i\gamma\varphi_2(z')}\right): \delta^{+}(z-z') \\ &+ :e^{2i\gamma\varphi_2(z)}e^{2i\gamma\varphi_2(z')}: \left(\frac{i\hbar}{2\gamma^2}(\eta^2-1)\partial_z\delta^{+}(z-z') + \hbar^2\left(\delta^{+}(z-z')\right)^2\right). \end{aligned} \tag{47}$$

Using the following identity

$$\left[\delta^{+}(z)\right]^{2}=\frac{1}{(4\pi)^{2}}+\frac{i}{2\pi}\partial_{z}\delta^{+}(z), \tag{48}$$

the right hand side of (47) can be written linearly in $\delta^{+}(z-z')$ and its derivative. This distribution becomes $\delta^{-}(z-z')$ on exchanging z and z'. Thus, if we take (47) and subtract the equation obtained by exchanging z and z', we get

$$\begin{aligned}\frac{W_{+}(z)W_{+}(z')}{e^{i\hbar\gamma^{2}h^{+}(z-z')}}-\frac{W_{+}(z')W_{+}(z)}{e^{-i\hbar\gamma^{2}h^{-}(z-z')}}=i\hbar\!:\!\left(e^{2i\gamma\varphi_{2}(z)}W_{+}(z')-W_{+}(z)e^{2i\gamma\varphi_{2}(z')}\right)\!:\!\delta_{2\pi}(z-z')\\+\frac{i\hbar}{2\gamma^{2}}\left(\eta^{2}-1+\frac{\hbar\gamma^{2}}{\pi}\right):e^{2i\gamma\varphi_{2}(z)}e^{2i\gamma\varphi_{2}(z')}:\partial_{z}\delta_{2\pi}(z-z').\end{aligned} \tag{49}$$

The first term on the right hand side is zero since the prefactor of the delta function tends to zero as $z\rightarrow z'$, and so (26) follows immediately. The other operator identities can be obtained in a similar manner.

4 Physical fields and metric

We now sketch the quantisation of the physical fields. Classically u has conformal weight zero

$$\{T(z),u(z',\bar{z}')\}=-\partial_{z'}u(z',\bar{z}')\delta_{2\pi}(z-z'). \tag{50}$$

One can also derive the following relation

$$\{V_{+}(z),u(z',\bar{z}')\}=\frac{i\gamma^{2}}{2}V_{+}(z)u(z',\bar{z}')\epsilon_{2\pi}(z-z'). \tag{51}$$

We propose to use (50) and (51) as a basis for the quantisation of u. The $\{V_{-},u\}$ bracket is more complicated (this does not reflect any disparity between V_{+} and V_{-} since $\{V_{-},\bar{u}\}$ is likewise simpler than $\{V_{+},\bar{u}\}$). Quantum mechanically we expect u to have a non-zero conformal weight and that the quantum version of (51) will involve the kind of deformed commutator considered in the previous section.

Finally, let us turn to the equal-time algebra of the physical fields. Classically we have the 'locality' relations

$$\{u(\sigma,\tau),u(\sigma',\tau)\}=\{u(\sigma,\tau),\bar{u}(\sigma',\tau)\}=\{\bar{u}(\sigma,\tau),\bar{u}(\sigma',\tau)\}=0, \tag{52}$$

whose quantum extensions are obvious. More interesting are the brackets

$$\{u(\sigma,\tau),\dot{\bar{u}}(\sigma',\tau)\}=\{\bar{u}(\sigma,\tau),\dot{u}(\sigma',\tau)\}=2\gamma^{2}(1+u\bar{u})\delta_{2\pi}(\sigma-\sigma'). \tag{53}$$

The coefficient of the delta is clearly related to the target space metric. Writing $ds^{2}=g_{\mu\nu}dx^{\mu}dx^{\nu}$, the coefficient can be identified with $g^{u\bar{u}}=g^{\bar{u}u}$. It would be interesting to compare the 'quantum metric' derived from the commutator $[u(\sigma,\tau),\dot{\bar{u}}(\sigma',\tau)]$ with the quantum deformations of the target space metric discussed in the literature [9, 10].

Acknowledgements

I am grateful to G. Jorjadze and G. Weigt for their vital input into the work reported here.

References

[1] E. Witten, Phys. Rev. **D44** 314 (1991)

[2] J.-L. Gervais and M.V. Saveliev, Phys. Lett. **B286** 271 (1992).

[3] A. Bilal, Nucl. Phys. **B422** 258 (1994).

[4] E. Braaten, T. Curtright, and C. Thorn, Ann. Phys. **147** 365 (1983)

[5] H.J. Otto and G. Weigt, Z. Phys.**C31** 219 (1986).

[6] U. Müller and G. Weigt, Comm. Math. Phys. **205** 421 (1999).

[7] C. Ford, J. Jorjadze and G. Weigt, *Integration of the $SL(2,R)/U(1)$ Gauged WZNW Theory by Reduction and Quantum Parafermions,* `hep-th/0003246.`

[8] C. Ford and L. O'Raifeartaigh, Nucl. Phys. **B460** 203 (1996).

[9] R. Dijkgraaf, E. Verlinde, H. Verlinde, Nucl. Phys. **B371** 269 (1992).

[10] A.A. Tseytlin, Nucl. Phys. **B399** 601 (1993).